Laplace Operator in Networks of Thin Fibers: Spectrum Near the Threshold.

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Abstract

Our talk at Lisbon SAMP conference was based mainly on our recent results on small diameter asymptotics for solutions of the Helmgoltz equation in networks of thin fibers. These results were published in [21]. The present paper contains a detailed review of [21] under some assumptions which make the results much more transparent. It also contains several new theorems on the structure of the spectrum near the threshold. small diameter asymptotics of the resolvent, and solutions of the evolution equation.

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1 Introduction

The paper concerns the asymptotic spectral analysis of the wave problems in systems of wave guides when the thickness ε of the wave guides is vanishing. In the simplest case, the problem is described by the stationary wave (Helmholtz) equation

$$-\varepsilon^2 \Delta u = \lambda u, \quad x \in \Omega_\varepsilon, \quad Bu = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{1}$$

in a domain $\Omega_\varepsilon \subset R^d$, $d \geq 2$, with infinitely smooth boundary (for simplicity) which has the following structure: Ω_ε is a union of a finite number of cylinders $C_{j,\varepsilon}$ (which we shall call channels), $1 \leq j \leq N$, of lengths l_j with the diameters of cross-sections of order $O(\varepsilon)$ and domains $J_{1,\varepsilon}, \cdots, J_{M,\varepsilon}$ (which we shall call junctions) connecting the channels into a network. It is assumed that the junctions have diameters of the same order $O(\varepsilon)$. The boundary condition has the form: $B = 1$ (the Dirichlet BC) or $B = \frac{\partial u}{\partial n}$ (the Neumann BC) or $B = \varepsilon \frac{\partial u}{\partial n} + \alpha(x)$, where n is the exterior normal and the function $\alpha > 0$ is real valued and does not depend on the longitudinal (parallel to the axis) coordinate on the boundary of the channels. One also can impose one type of BC on the lateral boundary of Ω_ε and another BC on free ends (which are not adjacent to a junction) of the channels. For simplicity we assume that only Dirichlet or Neumann BC are imposed on the free ends of the channels. Sometimes we shall denote the operator B on the lateral surface of Ω_ε by B_0, and we shall denote the operator B on the free ends of the channels by B_e.

Let m channels have infinite length. We start the numeration of $C_{j,\varepsilon}$ with the infinite channels. So, $l_j = \infty$ for $1 \leq j \leq m$. The axes of the channels form edges Γ_j, $1 \leq j \leq N$, of the limiting ($\varepsilon \to 0$) metric graph Γ. We split the set V of vertices v_j of the graph in two subsets $V = V_1 \cup V_2$, where the vertices from the set V_1 have degree 1 and the vertices from the set V_2 have degree at least
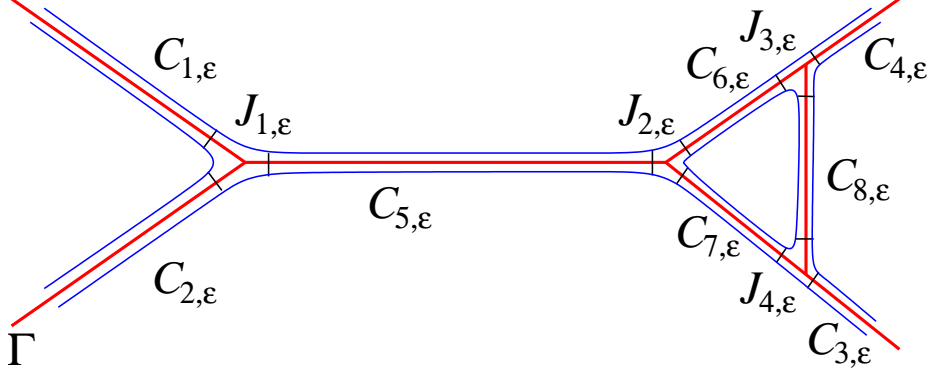
Figure 1: An example of a domain Ω_ε with four junctions, four unbounded channels and four bounded channels.

two, i.e. vertices $v_i \in V_1$ of the graph Γ correspond to the free ends of the channels, and vertices $v_j \in V_2$ correspond to the junctions $J_{j,\varepsilon}$.

Equation (1) degenerates when $\varepsilon = 0$. One could omit ε^2 in (1). However, the problem under consideration would remain singular, since the domain Ω_ε shrinks to the graph Γ as $\varepsilon \to 0$. The presence of this coefficient is convenient, since it makes the spectrum less vulnerable to changes in ε. As we shall see, in some important cases the spectrum of the problem does not depend on ε, and the spectrum will be proportional to ε^{-2} if ε^2 in (1) is omitted. The operator in $L^2(\Omega_\varepsilon)$ corresponding to the problem (1) will be denoted by H_ε.

The goal of this paper is the asymptotic analysis of the spectrum of H_ε, the resolvent $(H_\varepsilon - \lambda)^{-1}$, and solutions of the corresponding non-stationary problems for the heat and wave equations as $\varepsilon \to 0$. One can expect that H_ε is close (in some sense) to a one dimensional operator on the limiting graph Γ with appropriate gluing conditions at the vertices $v \in V$. The justification of this fact is not always simple. The form of the GC in general situation was discovered quite recently in our previous paper [21].

An important class of domains Ω_ε are self-similar domains with only one junction and all the channels being infinite. We shall call them *spider domains*. Thus, if Ω_ε is a spider domain, then there exist a point $\widehat{x} = x(\varepsilon)$ and an ε-independent domain Ω such that

$$\Omega_\varepsilon = \{(\widehat{x} + \varepsilon x) : x \in \Omega\}. \tag{2}$$

Thus, Ω_ε is the ε-contraction of $\Omega = \Omega_1$.

For the sake of simplicity we shall assume that Ω_ε is self-similar in a neighborhood of each junction. Namely, let $J_{j(v),\varepsilon}$ be the junction which corresponds to a vertex $v \in V$ of the limiting graph Γ. Consider a junction $J_{j(v),\varepsilon}$ and all the channels adjacent to $J_{j(v),\varepsilon}$. If some of these channels have finite length, we extend them to infinity. We assume that, for each $v \in V$, the resulting domain $\Omega_{v,\varepsilon}$ which consists of the junction $J_{j(v),\varepsilon}$ and the semi-infinite channels emanating from it is a spider domain. We also assume that all the channels $C_{j,\varepsilon}$ have the same cross-section ω_ε. This assumption is needed only to make the results more transparent. From the self-similarity assumption it follows that ω_ε is an ε−homothety of a bounded domain $\omega \in R^{d-1}$.

Let $\lambda_0 < \lambda_1 \le \lambda_2...$ be eigenvalues of the negative Laplacian $-\Delta_{d-1}$ in ω with the BC $B_0 u = 0$ on $\partial\omega$ where we put $\varepsilon = 1$ in B_0, and let $\{\varphi_n(y)\}, y \in \omega \in R^{d-1}$, be the set of corresponding orthonormal eigenfunctions. Then λ_n are eigenvalues of $-\varepsilon^2 \Delta_{d-1}$ in ω_ε and $\{\varphi_n(y/\varepsilon)\}$ are the corresponding eigenfunctions. In the presence of infinite channels, the spectrum of the operator H_ε consists of

an absolutely continuous component which coincides with the semi-bounded interval $[\lambda_0, \infty)$ and a discrete set of eigenvalues. The eigenvalues can be located below λ_0 and can be embedded into the absolutely continuous spectrum. We will call the point $\lambda = \lambda_0$ the threshold since it is the bottom of the absolutely continuous spectrum or (and) the first point of accumulation of the eigenvalues as $\varepsilon \to 0$. Let us consider two simplest examples: the Dirichlet problem in a half infinite cylinder and in a bounded cylinder of the length l. In the first case, the spectrum of the negative Dirichlet Laplacian in Ω_ε is pure absolutely continuous and has multiplicity $n+1$ on the interval $[\lambda_n, \infty)$. In the second case the spectrum consists of the set of eigenvalues $\lambda_{n,m} = \lambda_n + \varepsilon^2 m^2 / l^2$, $n \geq 0$, $m \geq 1$.

The wave propagation governed by the operator H_ε can be described in terms of the scattering solutions and scattering matrices associated to individual junctions of Ω_ε. The scattering solutions give information on the absolutely continuous spectrum and the resolvent for energies in the bulk of the spectrum ($\lambda > \lambda_0$). The spectrum in a small neighbourhood of λ_0 and below λ_0 is associated with the parabolic equation. However, the scattering solutions allow us to approximate the operator H_ε, $\varepsilon \to 0$, by a one dimensional operator on the limiting graph for all values of $\lambda \geq \lambda_0$. In particular, when $\lambda \approx \lambda_0$ the corresponding GC on the limiting graph are expressed in terms of the limits of the scattering matrices as $\lambda \to \lambda_0$.

The plan of the paper is as follows. Next section is devoted to historical remarks. A more detailed description of the results from our paper [21] on the asymptotic behavior of the scattering solutions ($\lambda > \lambda_0 + \delta$, $\varepsilon \to 0$) is given in section 3. In particular, the GC on the limiting graph are described ($\lambda > \lambda_0 + \delta$). The Green function of the one dimensional problem on the limiting graph is studied in section 4. The resolvent convergence as $\varepsilon \to 0$ is established in section 5 when λ is near λ_0. This allows us to derive and rigorously justify the GC for the limiting problem with $\lambda \approx \lambda_0$ which where obtained earlier [21] only on a formal level, as the limit as $\lambda \to \lambda_0$ of the GC with $\lambda > \lambda_0$. A detailed analysis of these GC is given.

Let $\lambda = \lambda_0 + O(\varepsilon^2)$. It has been known (see references in the next section) that for an arbitrary domain Ω_ε and the Neumann boundary condition on $\partial \Omega_\varepsilon$, the GC on the limiting graph Γ is Kirchhoff's condition. The GC are different for other boundary conditions on $\partial \Omega_\varepsilon$. It was shown in [21] that for generic domains Ω_ε and the boundary conditions different from the Neumann condition, the GC at the vertices of Γ are Dirichlet conditions. It is shown here that, for arbitrary domain Ω_ε, the GC at each vertex v of the limiting graph has the following form. For any function ς on Γ, we form a vector $\varsigma^{(v)}$ whose components are restrictions of ς to the edges of Γ adjacent to v. The GC at v, with λ near λ_0, are the Dirichlet condition for some components of the vector $\widetilde{\varsigma}^{(v)}$ and the Neumann condition for the remaining components where $\widetilde{\varsigma}^{(v)}$ is a rotation of $\varsigma^{(v)}$.

Note that the resolvent convergence provides the convergence of the discrete spectrum. In the presence of finite channels $C_{j,\varepsilon}$, the operator H_ε has a sequence of eigenvalues which converge to λ_0 as $\varepsilon \to 0$ (see the example above). Thus, these eigenvalues are asymptotically ($\varepsilon \to 0$) close to the eigenvalues of the problem where the junctions are replaced by Dirichlet/Neumann boundary conditions. The final result concerns the inverse scattering problem. The GC of the limiting problem depend on λ if $\lambda > \lambda_0 + \delta$. A λ-independent effective potential is constructed in the last section of the paper which has the same scattering data as the original problem. This allows one to reduce the problem in Ω_ε to a one dimensional problem with λ-independent GC.

2 Historical remarks.

Certain problems related to the operator H_ε have been studied in detail. They concern, directly or indirectly, the spectrum near the origin for the operator H_ε with the Neumann boundary condition on $\partial \Omega_\varepsilon$, see [5, 6, 12, 13, 15, 18, 19, 25, 26]. The following couple of features distinguish the Neumann boundary condition. First, only in this case the ground states $\varphi_0(y/\varepsilon) = 1$ on the cross sections of the channels can be extended smoothly onto the junctions (by 1) to provide the ground state for the operator H_ε in an arbitrary domain Ω_ε. Another important fact, which is valid only in the case

of the Neumann boundary conditions, is that $\lambda_0 = 0$. Note that an eigenvalue $\lambda = \mu$ of the operator H_ε contributes a term of order $e^{-\frac{\mu t}{\varepsilon^2}}$ to the solutions of the heat equation in Ω_ε. The existence of the spectrum in a small (of order $O(\varepsilon^2)$) neighborhood of the origin leads to the existence of a non-trivial limit, as $\varepsilon \to 0$, for the solutions of the heat equation. Solutions of the heat equation with other boundary conditions are vanishing exponentially as $\varepsilon \to 0$.

The GC and the justification of the limiting procedure $\varepsilon \to 0$ when λ is near $\lambda_0 = 0$ and the Neumann BC is imposed at the boundary of Ω_ε can be found in [12], [18], [19], [26]. Typically, the GC at the vertices of the limiting graph in this case are: the continuity at each vertex v of both the field and the flow. These GC are called Kirchhoff's GC. The paper [12] provides the convergence, as $\varepsilon \to 0$, of the Markov process on Ω_ε to the Markov process on the limiting graph for more general domains Ω_ε (the cross section of a channel can vary). In the case when the shrinkage rate of the volume of the junctions is lower than the one of the area of the cross-sections of the guides, more complex, energy dependent or decoupling, conditions may arise (see [15], [19], [5] for details).

The operator H_ε with the Dirichlet boundary condition on $\partial\Omega_\varepsilon$ was studied in a recent paper [24] under conditions that λ is near the threshold $\lambda_0 > 0$ and the junctions are more narrow than the channels. It is assumed there that the domain Ω_ε is bounded. Therefore, the spectrum of the operator (1) is discrete. It is proved that the eigenvalues of the operator (1) in a small neighborhood of λ_0 behave asymptotically, when $\varepsilon \to 0$, as eigenvalues of the problem in the disconnected domain that one gets by omitting the junctions, separating the channels in Ω_ε, and adding the Dirichlet conditions on the bottoms of the channels. This result indicates that the waves do not propagate through the narrow junctions when λ is close to the bottom of the absolutely continuous spectrum. A similar result was obtained in [2] for the Schrödinger operator with a potential having a deep strict minimum on the graph, when the width of the walls shrinks to zero. It will be shown in this paper, that the same result (the GC on the limiting graph is the Dirichlet condition if H_ε is the operator with the Dirichlet boundary condition on $\partial\Omega_\varepsilon$ and $\lambda \approx \lambda_0 > 0$) is valid for generic domains Ω_ε without assumptions on the size of the junctions.

The asymptotic analysis of the scattering solutions and the resolvent for operator H_ε with arbitrary boundary conditions on $\partial\Omega_\varepsilon$ and λ in the bulk of the absolutely continuous spectrum ($\lambda > \lambda_0 + \delta$) was given by us in [20], [21]. It was shown there that the GC on the limiting graph can be expressed in terms of the scattering matrices defined by junctions of Ω_ε. Formal extension of these conditions to $\lambda = \lambda_0$ leads to the Dirichlet boundary conditions at the vertices of the limiting graph for generic domains Ω_ε. Among other results, we will show here that the asymptotics obtained in [20], [21] are valid up to $\lambda = \lambda_0$.

There is extended literature on the spectrum of the operator H_ε below the threshold λ_0 (for example, see [3]-[11] and references therein). We shall not discuss this topic in the present paper. Important facts on the scattering solutions in networks of thin fibers can be found in [23], [22].

3   Scattering solutions.

We introduce Euclidean coordinates (t, y) in channels $C_{j,\varepsilon}$ chosen in such a way that t-axis is parallel to the axis of the channel, hyperplane R_y^{d-1} is orthogonal to the axis, and $C_{j,\varepsilon}$ has the following form in the new coordinates:

$$C_{j,\varepsilon} = \{(t, \varepsilon y) : 0 < t < l_j, \ y \in \omega\}.$$

Let us recall the definition of scattering solutions for the problem in Ω_ε. In this paper, we'll need the scattering solutions only in the case of $\lambda \in (\lambda_0, \lambda_1)$. Consider the non-homogeneous problem

$$(-\varepsilon^2\Delta - \lambda)u = f, \quad x \in \Omega_\varepsilon; \quad Bu = 0 \ \text{on} \ \partial\Omega_\varepsilon. \tag{3}$$

4

Definition 1 *Let $f \in L^2_{com}(\Omega_\varepsilon)$ have a compact support, and $\lambda_0 < \lambda < \lambda_1$. A solution u of (3) is called outgoing if it has the following asymptotic behavior at infinity in each infinite channel $C_{j,\varepsilon}$, $1 \leq j \leq m$:*

$$u = a_j e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}\varphi_0(y/\varepsilon) + O(e^{-\gamma t}), \quad \gamma = \gamma(\varepsilon, \lambda) > 0. \tag{4}$$

If $\lambda < \lambda_0$, a solution u of (3) is called outgoing if it decays at infinity.

Definition 2 *Let $\lambda_0 < \lambda < \lambda_1$. A function $\Psi = \Psi_p^{(\varepsilon)}$, $0 \leq p \leq m$, is called a solution of the scattering problem in Ω_ε if*

$$(-\Delta - \lambda)\Psi = 0, \quad x \in \Omega_\varepsilon; \quad B\Psi = 0 \quad on \ \partial\Omega_\varepsilon, \tag{5}$$

and Ψ has the following asymptotic behavior in each infinite channel $C_{j,\varepsilon}$, $1 \leq j \leq m$:

$$\Psi_p^{(\varepsilon)} = \delta_{p,j}e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}\varphi_0(y/\varepsilon) + t_{p,j}e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}\varphi_0(y/\varepsilon) + O(e^{-\gamma t}), \tag{6}$$

where $\gamma = \gamma(\varepsilon, \lambda) > 0$, and $\delta_{p,j}$ is the Kronecker symbol, i.e. $\delta_{p,j} = 1$ if $p = j$, $\delta_{p,j} = 0$ if $p \neq j$.

The first term in (6) corresponds to the incident wave (coming through the channel $C_{p,\varepsilon}$), and all the other terms describe the transmitted waves. The transmission coefficients $t_{p,j} = t_{p,j}(\varepsilon, \lambda)$ depend on ε and λ. The matrix

$$T = [t_{p,j}] \tag{7}$$

is called the scattering matrix.

The outgoing and scattering solutions are defined similarly when $\lambda \in (\lambda_n, \lambda_{n+1})$. In this case, any outgoing solution has $n + 1$ waves in each channel propagating to infinity with the frequencies $\sqrt{\lambda - \lambda_s}/\varepsilon$, $0 \leq s \leq n$. There are $m(n+1)$ scattering solutions: the incident wave may come through one of m infinite channels with one of $(n + 1)$ possible frequencies. The scattering matrix has the size $m(n + 1) \times m(n + 1)$ in this case.

Theorem 3 *The scattering matrix T, $\lambda > \lambda_0$, $\lambda \notin \{\lambda_j\}$, is unitary and symmetric ($t_{p,j} = t_{j,p}$).*

The operator H_ε is non-negative, and therefore the resolvent

$$R_\lambda = (H_\varepsilon - \lambda)^{-1} : L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon) \tag{8}$$

is analytic in the complex λ plane outside the positive semi-axis $\lambda \geq 0$. Hence, the operator R_{k^2} is analytic in k in the half plane $\text{Im}\,k > 0$. We are going to consider the analytic extension of the operator R_{k^2} to the real axis and the lower half plane. Such an extension does not exist if R_{k^2} is considered as an operator in $L^2(\Omega_\varepsilon)$ since R_{k^2} is an unbounded operator when $\lambda = k^2$ belongs to the spectrum of the operator R_λ. However, one can extend R_{k^2} analytically if it is considered as an operator in the following spaces (with a smaller domain and a larger range):

$$R_{k^2} : L^2_{com}(\Omega_\varepsilon) \to L^2_{loc}(\Omega_\varepsilon). \tag{9}$$

Theorem 4 *(1) The spectrum of the operator H_ε consists of the absolutely continuous component $[\lambda_0, \infty)$ (if Ω_ε has at least one infinite channel) and, possibly, a discrete set of positive eigenvalues $\lambda = \mu_{j,\varepsilon}$ with the only possible limiting point at infinity. The multiplicity of the a.c. spectrum changes at points $\lambda = \lambda_n$, and is equal to $m(n + 1)$ on the interval $(\lambda_n, \lambda_{n+1})$.*

If Ω_ε is a spider domain, then the eigenvalues $\mu_{j,\varepsilon} = \mu_j$ do not depend on ε.

(2) The operator (9) admits a meromorphic extension from the upper half plane $\text{Im}\,k > 0$ into lower half plane $\text{Im}\,k < 0$ with the branch points at $k = \pm\sqrt{\lambda_n}$ of the second order and the real poles at $k = \pm\sqrt{\mu_{j,\varepsilon}}$ and, perhaps, at some of the branch points (see the remark below). The resolvent

5

(9) has a pole at $k = \pm\sqrt{\lambda_n}$ if and only if the homogeneous problem (3) with $\lambda = \lambda$ has a nontrivial solution u such that

$$u = a_j\varphi_n(y/\varepsilon) + (e^{-\gamma t}), \quad x \in C_{j,\varepsilon}, \quad t \to \infty, \quad 1 \le j \le m. \tag{10}$$

(3) If $f \in L^2_{com}(\Omega_\varepsilon)$, and $k = \sqrt{\lambda}$ is real and is not a pole or a branch point of the operator (9), and $\lambda > \lambda_0$, then the problem (3), (4) is uniquely solvable and the outgoing solution u can be found as the $L^2_{loc}(\Omega_\varepsilon)$ limit

$$u = R_{\lambda+i0}f. \tag{11}$$

(4) There exist exactly $m(n+1)$ different scattering solutions for the values of $\lambda \in (\lambda_n, \lambda_{n+1})$ such that $k = \sqrt{\lambda}$ is not a pole of the operator (9), and the scattering solution is defined uniquely after the incident wave is chosen.

Remark. The pole of R_λ at a branch point $\lambda = \lambda_n$ is defined as the pole of this operator function considered as a function of $z = \sqrt{\lambda - \lambda_n}$.

Let us describe the asymptotic behavior of scattering solutions $\Psi = \Psi_p^{(\varepsilon)}$ as $\varepsilon \to 0$, $\lambda \in (\lambda_0, \lambda_1)$. We shall consider here only the first zone of the absolutely continuous spectrum, but one can find the asymptotics of $\Psi_p^{(\varepsilon)}$ in [21] for any $\lambda > \lambda_0$. Note that an arbitrary solution u of equation (1) in a channel $C_{j,\varepsilon}$ can be represented as a series with respect to the orthogonal basis $\{\varphi_n(y/\varepsilon)\}$ of the eigenfunctions of the Laplacian in the cross-section of $C_{j,\varepsilon}$. Thus, it can be represented as a linear combination of the travelling waves

$$e^{\pm i\frac{\sqrt{\lambda-\lambda_n}}{\varepsilon}t}\varphi_n(y/\varepsilon), \quad \lambda \in (\lambda_n, \lambda_{n+1}),$$

and terms which grow or decay exponentially along the axis of $C_{j,\varepsilon}$. The main term of small ε asymptotics of scattering solutions contains only travelling waves, i.e. functions $\Psi_p^{(\varepsilon)}$ in each channel $C_{j,\varepsilon}$ have the following form when $\lambda \in (\lambda_0, \lambda_1)$:

$$\Psi = \Psi_p^{(\varepsilon)} = (\alpha_{p,j}e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + \beta_{p,j}e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t})\varphi_0(y/\varepsilon) + r_{p,j}^\varepsilon, \quad x \in C_{j,\varepsilon}, \tag{12}$$

where

$$|r_{p,j}^\varepsilon| \le Ce^{-\frac{\gamma d(t)}{\varepsilon}}, \quad \gamma > 0, \quad \text{and} \quad d(t) = \min(t, l_j - t).$$

The constants $\alpha_{p,j}$, $\beta_{p,j}$ and functions $r_{p,j}^\varepsilon$ depend on λ and ε. Formula (12) can be written as follows

$$\Psi = \Psi_p^{(\varepsilon)} = \varsigma\varphi_0(y/\varepsilon) + r_p^\varepsilon, \quad |r_p^\varepsilon| \le Ce^{-\frac{\gamma d(t)}{\varepsilon}}, \tag{13}$$

where the function $\varsigma = \varsigma(t)$ can be considered as a function on the limiting graph Γ which is equal to $\varsigma_j(t), 0 < t < l_j$, on the edge Γ_j and satisfies the following equation:

$$(\varepsilon^2\frac{d^2}{dt^2} + \lambda - \lambda_0)\varsigma = 0. \tag{14}$$

In order to complete the description of the main term of the asymptotic expansion (12), we need to provide the choice of constants in the representation of ς_j as a linear combinations of the exponents. We specify ς by imposing conditions at infinity and gluing conditions (GC) at each vertex v of the graph Γ. Let $V = \{v\}$ be the set of vertices v of the limiting graph Γ. These vertices correspond to the free ends of the channels and the junctions in Ω_ε.

The conditions at infinity concern only the infinite channels $C_{j,\varepsilon}, j \le m$. They indicate that the incident wave comes through the channel $C_{p,\varepsilon}$. They have the form:

$$\beta_{p,j} = \delta_{p,j}. \tag{15}$$

6

The GC at vertices v of the graph Γ are universal for all incident waves and depend on λ. We split the set V of vertices v of the graph in two subsets $V = V_1 \cup V_2$, where the vertices from the set V_1 have degree 1 and correspond to the free ends of the channels, and the vertices from the set V_2 have degree at least two and correspond to the junctions $J_{j,\varepsilon}$. We keep the same BC at $v \in V_1$ as at the free end of the corresponding channel of Ω_ε :

$$B_e \zeta = 0 \quad \text{at } v \in V_1. \tag{16}$$

In order to state the GC at a vertex $v \in V_2$, we choose the parametrization on Γ in such a way that $t = 0$ at v for all edges adjacent to this particular vertex. The origin ($t = 0$) on all the other edges can be chosen at any of the end points of the edge. Let $d = d(v) \geq 2$ be the order (the number of adjacent edges) of the vertex $v \in V_2$. For any function ς on Γ, we form a vector $\varsigma^{(v)} = \varsigma^{(v)}(t)$ with $d(v)$ components equal to the restrictions of ς on the edges of Γ adjacent to v. We shall need this vector only for small values of $t \geq 0$. Consider auxiliary scattering problems for the spider domain $\Omega_{v,\varepsilon}$. The domain is formed by the individual junction which corresponds to the vertex v, and all channels with an end at this junction, where the channels are extended to infinity if they have a finite length. We enumerate the channels of $\Omega_{v,\varepsilon}$ according to the order of the components of the vector $\varsigma^{(v)}$. We denote by Γ_v the limiting graph defined by $\Omega_{v,\varepsilon}$. Definitions 1, 2 and Theorem 4 remain valid for the domain $\Omega_{v,\varepsilon}$. In particular, one can define the scattering matrix $T = T_v(\lambda)$ for the problem (1) in the domain $\Omega_{v,\varepsilon}$. Let I_v be the unit matrix of the same size as the size of the matrix $D_v(\lambda)$. The GC at the vertex $v \in V_2$ has the form

$$i\varepsilon[I_v + T_v(\lambda)] \frac{d}{dt} \varsigma^{(v)}(t) - \sqrt{\lambda - \lambda_0}[I_v - T_v(\lambda)]\varsigma^{(v)}(t) = 0, \quad t = 0, \quad v \in V_2. \tag{17}$$

One has to keep in mind that the self-similarity of the spider domain $\Omega_{v,\varepsilon}$ implies that $T_v = T_v(\lambda)$ does not depend on ε.

Definition 5 *A family of subsets $l(\varepsilon)$ of a bounded closed interval $l \subset R^1$ will be called thin if, for any $\delta > 0$, there exist constants $\beta > 0$ and c_1, independent of δ and ε, and $c_2 = c_2(\delta)$, such that $l(\varepsilon)$ can be covered by c_1 intervals of length δ together with $c_2\varepsilon^{-1}$ intervals of length $c_2 e^{-\beta/\varepsilon}$. Note that $|l(\varepsilon)| \to 0$ as $\varepsilon \to 0$.*

Theorem 6 *For any bounded closed interval $l \subset (\lambda_0, \lambda_1)$, there exists $\gamma = \gamma(l) > 0$ and a thin family of sets $l(\varepsilon)$ such that the asymptotic expansion (13) holds on all (finite and infinite) channels $C_{j,\varepsilon}$ uniformly in $\lambda \in l \setminus l(\varepsilon)$ and x in any bounded region of R^d. The function ς in (13) is a vector function on the limiting graph which satisfies the equation (14), conditions (15) at infinity, BC (16), and the GC (17).*

    Remarks. 1) For spider domains, the estimate of the remainder is uniform for all $x \in R^d$.

    2) The asymptotics stated in Theorem 6 is valid only outside of a thin set $l(\varepsilon)$ since the poles of resolvent (8) may run over the interval (λ_0, λ_1) as $\varepsilon \to 0$, and the scattering solution may not exist when λ is a pole of the resolvent. These poles do not depend on ε for spider domains, and the set $l(\varepsilon)$ is ε-independent in this case.

    Consider a spider domain $\Omega_{v,\varepsilon}$ and scattering solutions $\Psi = \Psi_p^{(\varepsilon)}$ in $\Omega_{v,\varepsilon}$ when λ belongs to a small neighborhood of λ_0, i.e.

$$\Psi = \Psi_{p,v}^{(\varepsilon)} = (\delta_{p,j} e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + t_{p,j}(\lambda)e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t})\varphi_0(y/\varepsilon) + r_{p,j}^\varepsilon, \quad x \in C_{j,\varepsilon}, \quad |\lambda - \lambda_0| \leq \delta, \tag{18}$$

where $r_{p,j}^\varepsilon$ decays exponentially as $t \to \infty$ and $t = t(x)$ is the coordinate of the point x. We define these solutions for all complex λ in the circle $|\lambda - \lambda_0| \leq \delta$ by the asymptotic expansion (18) when $\text{Im}\lambda \geq 0, \lambda \neq \lambda_0$, and by extending them analytically for other values of λ in the circle.

7

Lemma 7 *Let $\Omega_{v,\varepsilon}$ be a spider domain. Then there exist δ and $\gamma > 0$ such that*
1) for each p and $|\lambda - \lambda_0| \le \delta$, $\lambda \ne \lambda_0$, the scattering solution exists and is unique,
2) the scattering coefficients $t_{p,j}(\lambda)$ are analytic in $\sqrt{\lambda - \lambda_0}$ when $|\lambda - \lambda_0| \le \delta$,
3) the following estimate is valid for the remainder

$$|r_{p,j}^{\varepsilon}| \le \frac{C}{|\lambda - \lambda_0|} e^{-\frac{\gamma t}{\varepsilon}}, \quad \gamma > 0, \quad x \in C_{j,\varepsilon}.$$

This statement can be extracted from the text of our paper [21]. Since it was not stated explicitly, we shall derive it from the theorems above. In fact, since the spider domain $\Omega_{v,\varepsilon}$ is self-similar, it is enough to prove this lemma when $\varepsilon = 1$. We omit index ε in Ω_ε, $C_{p,\varepsilon}$, $\Psi_p^{(\varepsilon)}$ when the problem in Ω_ε is considered with $\varepsilon = 1$. Let $\alpha_p(x)$ be a C^∞- function on Ω which is equal to zero outside of the channel C_p and equal to one on C_p when $t > 1$. We look for the solution Ψ_p of the scattering problem in the form

$$\Psi_p = \delta_{p,j} e^{-i\sqrt{\lambda - \lambda_0} t} \varphi_0(y) \alpha_p(x) + u_p. \qquad \Im\lambda \ge 0, \ \lambda \ne \lambda_0,$$

Then u_p is the outgoing solution of the problem

$$(-\Delta - \lambda)u = f, \quad x \in \Omega; \quad u = 0 \ \text{ on } \partial\Omega,$$

where
$$f = -\delta_{p,j}[2\nabla(e^{i\sqrt{\lambda - \lambda_0} t} \varphi_0(y))\nabla\alpha_p(x) + e^{i\sqrt{\lambda - \lambda_0} t} \varphi_0(y)\Delta\alpha_p(x)] \in L^2_{com}(\Omega).$$

From Theorem 4 it follows that there exists $\delta > 0$ such that u_p exists and is unique when $|\lambda - \lambda_0| \le \delta$, $\Im\lambda \ge 0$, $\lambda \ne \lambda_0$, and $u_p = R_\lambda f$ when $\Im\lambda \ge 0$, and u_p can be extended analytically to the lower half-plane if R_λ is understood as in (9). The function $u_p = R_\lambda f$ may have a pole at $\lambda = \lambda_0$. In particular, on the cross-sections $t = 1$ of the infinite channels C_j, the function u_p is analytic in $\sqrt{\lambda - \lambda_0}$ (when $|\lambda - \lambda_0| \le \delta$) with a possible pole at $\lambda = \lambda_0$. We note that $f = 0$ on $C_j \cap \{t \ge 1\}$. We represent u_p there as a series with respect to the basis $\{\varphi_s(y)\}$. This leads to (18) and justifies all the statements of the lemma if we take into account the following two facts: 1) the resolvent (9) can not have a singularity at $\lambda = \lambda_0$ of order higher than $1/|\lambda - \lambda_0|$, since the norm of the resolvent (8) at any point λ does not exceed the inverse distance from λ to the spectrum, 2) the scattering coefficients can not have a singularity at $\lambda = \lambda_0$ due to Theorem 3.

The proof of Lemma 7 is complete.

4 Spectrum of the problem on the limiting graph.

Let us write the inhomogeneous problem on the limiting graph Γ which corresponds to the scattering problem (14), (15), (16), (17). We shall always assume that the function f in the right-hand side in the equation below has compact support. Then the corresponding inhomogeneous problem has the form

$$(\varepsilon^2 \frac{d^2}{dt^2} + \lambda - \lambda_0)\varsigma = f \ \text{ on } \Gamma, \tag{19}$$

$$B_e\varsigma = 0 \ \text{ at } v \in V_1, \qquad i\varepsilon[I_v + T_v(\lambda)]\frac{d}{dt}\varsigma^{(v)}(0) - \sqrt{\lambda - \lambda_0}[I_v - T_v(\lambda)]\varsigma^{(v)}(0) = 0 \qquad \text{at } v \in V_2, \tag{20}$$

$$\varsigma = \beta_j e^{i\frac{\sqrt{\lambda - \lambda_0}}{\varepsilon} t}, \quad t >> 1, \ \text{ on the infinite edges } \Gamma_j, \ 1 \le j \le m. \tag{21}$$

This problem is relevant to the original problem in Ω_ε only while $\lambda < \lambda_1$, since more than one mode in each channel survives as $\varepsilon \to 0$ when $\lambda > \lambda_1$. The latter leads to a more complicated

problem on the limiting graph (see [21]). We are going to use the problem (19)-(21) to study the spectrum of the operator H_ε when

$$\lambda = \lambda_0 + \varepsilon^2 \mu, \quad |\mu| < c. \tag{22}$$

As we shall see later, if Ω_ε has a channel of finite length, then the operator H_ε has a sequence of eigenvalues which are at a distance of order $O(\varepsilon^2)$ from the threshold $\lambda = \lambda_0$. For example, if Ω_ε is a finite cylinder with the Dirichlet boundary condition (see the introduction) these eigenvalues have the form $\lambda_0 + \varepsilon^2 m^2/l^2$, $m \geq 1$. Thus, assumption (22) allows one to study any finite number of eigenvalues near $\lambda = \lambda_0$.

Let us make a substitution $\lambda = \lambda_0 + \varepsilon^2 \mu$ in (19)-(21). Condition (20) may degenerate at $\mu = 0$, and one needs to understand this condition at $\mu = 0$ as the limit when $\mu \to 0$ after an appropriate normalization which will be discussed later.

Lemma 8 . *There is an orthogonal projection $P = P(v)$ in $R^{d(v)}$ such that the problem (19)-(21) under condition (22) can be written in the form*

$$(\frac{d^2}{dt^2} + \mu)\varsigma = \varepsilon^{-2} f \quad on \ \Gamma; \tag{23}$$

$$B_e \varsigma = 0 \quad at \ v \in V_1; \qquad P\varsigma^{(v)}(0) + O(\varepsilon)\frac{d}{dt}\varsigma^{(v)}(0) = 0, \quad P^\perp \frac{d}{dt}\varsigma^{(v)}(0) + O(\varepsilon)\varsigma^{(v)}(0) = 0 \qquad at \ v \in V_2, \tag{24}$$

$$\varsigma = \beta_j e^{i\sqrt{\mu}t}, \quad t >> 1, \quad on \ the \ infinite \ edges \ \Gamma_{j,} \quad 1 \leq j \leq m, \tag{25}$$

where $d(v)$ is the order (the number of adjacent edges) of the vertex v and $P^\perp$ is the orthogonal complement to P.

Remarks. 1) The GC (23) at $v \in V_2$ looks particularly simple in the eigenbasis of the operators $P, P^\perp$. If $\varepsilon = 0$, then it is the Dirichlet/Neumann GC, i.e after appropriate orthogonal transformation $\xi^v = C_v \tilde{\xi}^v$,

$$\tilde{\xi}_1^v(0) = \cdots = \tilde{\xi}_k^v(0) = 0, \ \frac{d\tilde{\xi}_{k+1}^v}{dt}(0) = \cdots = \frac{d\tilde{\xi}_d^v}{dt}(0) = 0, \quad k = \text{Rank}P.$$

2) We consider μ as being a spectral parameter of the problem (23)-(25), but one needs to keep in mind that the terms $O(\varepsilon)$ in condition (23) depend on μ.

Proof. Let us recall that the matrix $T_v(\lambda)$ is analytic in $\sqrt{\lambda - \lambda_0}$ due to Lemma 7. Theorem 3 implies the existence of the orthogonal matrix C_v such that $D_v := C_v^{-1}T(\lambda_0)C_v$ is a diagonal matrix with elements $\nu_s = \pm 1, 1 \leq s \leq d(v)$, on the diagonal. In fact, from Theorem 3 it follows that, for any $\lambda \in [\lambda_0, \lambda_1]$, one can reduce $T(\lambda)$ to a diagonal form with diagonal elements $\nu_s = \nu_s(\lambda)$ where $|\nu_s| = 1$. Additionally, one can easily show that the matrix $T(\lambda_0)$ is real-valued, and therefore, $\nu_s = \pm 1$ when $\lambda = \lambda_0$. The statement of the lemma follows immediately from here with $P = \frac{1}{2}(I - D_v)C_v^{-1}$.

Consider the Green function $G_\varepsilon(\gamma, \gamma_0, \mu)$, $\gamma, \gamma_0 \in \Gamma$, $\gamma_0 \notin V$, of the problem (23)-(25) which is the solution of the problem with $\varepsilon^{-2} f$ replaced by $\delta_{\gamma_0}(\gamma)$. Here $\delta_{\gamma_0}(\gamma)$ is the delta function on Γ supported at the point γ_0 which belongs to one of the edges of Γ (γ_0 is not a vertex). The Green function $G_0(\gamma, \gamma_0, \mu)$ is the solution of (23)-(25) with $\varepsilon = 0$ in (24).

Let us denote by P_0 the closure in $L^2(\Gamma)$ of the operator $-\frac{d^2}{dt^2}$ defined on smooth functions satisfying (24) with $\varepsilon = 0$. Note that the conditions (24) with $\varepsilon = 0$ do not depend on μ. Hence, P_0 is a self-adjoint operator whose spectrum consists of an absolutely continuous component $\{\mu \geq 0\}$ (if Γ has at least one unbounded edge) and a discrete set $\{\mu_j\}$ of non-negative eigenvalues. Let us denote by B_c the disk $|\mu| < c$ of the complex μ-plane.

9

Lemma 9 *For any $c > 0$ there exist $\varepsilon_0 > 0$ such that*

1) the eigenvalues $\{\mu_s(\varepsilon)\}$ of the problem (23)-(25) in the disk B_c of the complex μ-plane are located in $C_1\varepsilon$-neighborhoods of the points $\{\mu_j\}$, $C_1 = C_1(c)$, and each such neighborhood contains p eigenvalues $\mu_s(\varepsilon)$ with multiplicity taken into account where p is the multiplicity of the eigenvalue μ_j,

2) the Green function G_ε exists and is unique when $\mu \in B_c \backslash \{\mu_s(\varepsilon)\}$ and has the form

$$G_\varepsilon(\gamma, \gamma_0, \mu) = \frac{g(\gamma, \gamma_0, \mu, \varepsilon)}{h(\mu, \varepsilon)},$$

where g is a continuous function of $\gamma \in \Gamma_m$, $\gamma_0 \in \Gamma_n$, $\mu \in B_c$, $\varepsilon \in [0, \varepsilon_0]$, functions g and h are analytic in $\sqrt{\mu}$ and ε, and d has zeros in B_c only at points $\mu = \mu_s(\varepsilon)$. Here Γ_m, Γ_n are arbitrary edges of Γ.

Proof. We denote by $t = t(\gamma)$ ($t_0 = t_0(\gamma_0)$) the value of the parameter on the edge of Γ which corresponds to the point $\gamma \in \Gamma_m$ ($\gamma_0 \in \Gamma_n$, respectively). We look for the Green function in the form

$$G_\varepsilon = \delta_{m,n} \frac{e^{i\sqrt{\mu}|t-t_0|}}{2i\sqrt{\mu}} + a_{m,n}e^{i\sqrt{\mu}t} + b_{m,n}e^{-i\sqrt{\mu}t}, \qquad \gamma \in \Gamma_m, \ \gamma_0 \in \Gamma_n, \tag{26}$$

where $\delta_{m,n}$ is the Kronecker symbol, and the functions $a_{m,n}$, $b_{m,n}$ depend on t_0, μ, ε. Obviously, (23) with $\varepsilon^{-2}f$ replaced by $\delta_{\gamma_0}(\gamma)$ holds. Let us fix the edge Γ_n which contains γ_0. We substitute (26) into (24), (25) and get $2N$ equations for $2N$ unknowns $a_{m,n}$, $b_{m,n}$, $1 \leq m \leq N$, n is fixed. The matrix M of this system depends analytically on $\sqrt{\mu}$, $\mu \in B_c$ and $\varepsilon \in [0, \varepsilon_0]$. The right-hand side has the form $c_1 e^{i\sqrt{\mu}t_0} + c_2 e^{-i\sqrt{\mu}t_0}$, where the vectors c_1, c_2 depend analytically on $\sqrt{\mu}$ and ε. This implies all the statements of the lemma if we take into account that the determinant of M with $\varepsilon = 0$ has zeroes at eigenvalues of the operator P_0. The proof is complete.

In order to justify the resolvent convergence of the operator H_ε as $\varepsilon \to 0$ and obtain the asymptotic behavior of the eigenvalues of the problem (1) near λ_0 we need to represent the Green function G_ε of the problem on the graph in a special form. We fix points γ_i strictly inside of the edges Γ_i. These points split Γ into graphs Γ_v^{cut} which consist of one vertex v and parts of adjacent edges up to corresponding points γ_i. If Γ_v is the limiting graph which corresponds to the spider domain $\Omega_{v,\varepsilon}$, then Γ_v^{cut} is obtained from Γ_v by cutting its edges at points γ_i.

When $\varepsilon \geq 0$ is small enough, equation (23) on Γ_v has $d(v)$ linearly independent solutions satisfying the condition from (24) which corresponds to the chosen vertex v. This is obvious if $\varepsilon = 0$ (when the components of the vector $C_v^{-1}\varsigma^{(v)}$ satisfy either the Dirichlet or the Neumann conditions at v). Therefore it is also true for small $\varepsilon \geq 0$. We denote this solution space by S_v. Let us fix a specific basis $\{\psi_{p,v}(\gamma, \mu, \varepsilon)\}$, $1 \leq p \leq d(v)$, in S_v. It is defined as follows. Let us change the numeration of the edges of Γ (if needed) in such a way that the first $d(v)$ edges are adjacent to v. We also choose the parametrization on these edges in such a way that $t = 0$ corresponds to v. Then

$$\psi_{p,v} = \delta_{p,j}e^{-i\sqrt{\mu}t} + t_{p,j}(\lambda)e^{i\sqrt{\mu}t}, \qquad \gamma \in \Gamma_j.$$

Here $t = t(\gamma)$, $\lambda = \lambda_0 + \varepsilon^2\mu$, $t_{p,j} = t_{p,j}^{(v)}$ are the scattering coefficients for the spider domain $\Omega_{v,\varepsilon}$. Obviously, $\psi_{p,v}$ satisfies conditions (24), and formula (18) can be written as

$$\Psi_{p,v}^{(\varepsilon)} = \psi_{p,v}(\gamma, \mu, \varepsilon)\varphi_0(y/\varepsilon) + r_{p,j}^\varepsilon, \qquad x \in C_{j,\varepsilon}, \quad \gamma = \gamma(x). \tag{27}$$

where $\gamma(x) \in \Gamma_j$ is defined by the cross-section of the channel $C_{j,\varepsilon}$ through the point x.

We shall choose one of the points γ_j in a special way. Namely, if $\gamma_0 \in \Gamma_n$ then we chose $\gamma_n = \gamma_0$. Then G_ε belongs to the solution space S_v, and from Lemma 9 we get

Lemma 10 *The Green function G_ε can be represented on each part Γ_v^{cut} of the graph Γ in the form*

$$G_\varepsilon(\gamma, \gamma_0, \mu) = \frac{1}{h(\mu, \varepsilon)} \sum_{1 \le p \le d(v)} a_{p,v} \psi_{p,v}(\gamma, \mu, \varepsilon), \qquad \mu \in B_c, \ \varepsilon \in [0, \varepsilon_0], \tag{28}$$

where the function h is defined in Lemma 9 and $a_{p,v} = a_{p,v}(\gamma_0, \mu, \varepsilon)$ are continuous functions which are analytic in ε and $\sqrt{\mu}$

5 Resolvent convergence of the operator H_ε.

We are going to study the asymptotic behavior of the resolvent $R_{\lambda, \varepsilon} = (H_\varepsilon - \lambda)^{-1}$ of the operator H_ε when (22) holds and $\varepsilon \to 0$. When μ is complex, the resolvent R_λ is understood in the sense of analytic continuation described in Theorem 4. In fact, we shall study $R_\lambda f$ only inside of the channels $C_{j,\varepsilon}$ and under the assumption that the support of f belongs to a bounded region inside of the channels. We fix finite segments $\Gamma_j' \subset \Gamma_j$ of the edges of the graph large enough to contain the points γ_j. Let $\Gamma' = \cup \Gamma_j'$. We denote by $C_\varepsilon' = \cup C_{j,\varepsilon}'$ the union of the finite parts $C_{j,\varepsilon}'$ of the channels which shrink to Γ_j' as $\varepsilon \to 0$. We shall identify functions from $L^2(C_\varepsilon')$ with functions from $L^2(\Omega_\varepsilon)$ equal to zero outside C_ε'. We also omit the restriction operator when functions on Ω_ε are considered only on C_ε'.

If $f \in L^2(C_\varepsilon')$, denote

$$\widehat{f}(\gamma, \varepsilon) = \frac{< f, \varphi_0(y/\varepsilon) >}{\|\varphi_0(y/\varepsilon)\|_{L^2}} = \int_{\Omega_\varepsilon} f \varphi_0(y/\varepsilon) dy / \|\varphi_0(y/\varepsilon)\|_{L^2}, \quad \gamma \in \Gamma.$$

We shall use the notation G_ε for the integral operator

$$G_\varepsilon \zeta(\gamma) = \int_\Gamma G_\varepsilon(\gamma, \gamma_0, \mu) \zeta(\gamma_0) d\gamma_0, \quad \gamma \in \Gamma.$$

Theorem 11 *Let (22) hold. Then for any disk B_c, there exist $\varepsilon_0 = \varepsilon_0(c)$ and a constant $C < \infty$ such that the function*

$$R_{\lambda,,\varepsilon} f = (H_\varepsilon - \lambda)^{-1} f, \quad f \in L^2(C_\varepsilon'),$$

is analytic in $\sqrt{\mu}$ when $\mu \in B_c \backslash O(\varepsilon)$, where $O(\varepsilon)$ is $C\varepsilon$-neighborhood of the set $\{\mu_j\}$, and has the form

$$\|R_{\lambda,\varepsilon} f - \varphi_0(y/\varepsilon) G_\varepsilon \widehat{f}(\gamma, \varepsilon)\|_{L^2(C_\varepsilon')} \le C\varepsilon \|f\|_{L^2(C_\varepsilon')}.$$

Remarks. 1) The points μ_j were introduced above as eigenvalues of the problem (23)-(25) on the graph with $\varepsilon = 0$. The GC in this case are the Dirichlet and Neumann conditions for the components of the vector $C_v^{-1} \varsigma^{(v)}$. Obviously, these points are also eigenvalues of the operator H_ε with the junctions of Ω_ε replaced by the same Dirichlet/Neumann conditions on the edges of the channels adjacent to the junctions.

2) The resolvent convergence stated in the theorem implies the convergence, as $\varepsilon \to 0$, of eigenvalues of operator H_ε to $\{\mu_j\}$. We could not guarantee the fact that the eigenvalues of the problem on the graph are real (see Lemma 9). Of course, they are real for operator H_ε.

Proof. We construct an approximation $K_{\lambda,\varepsilon}$ to the resolvent $R_{\lambda,\varepsilon} = (H_\varepsilon - \lambda)^{-1}$ for $f \in L^2(C_\varepsilon')$. We represent $L^2(C_\varepsilon')$ as the orthogonal sum

$$L^2(C_\varepsilon') = L_0^2(C_\varepsilon') + L_1^2(C_\varepsilon'),$$

where functions from $L_0^2(C_\varepsilon')$ have the form $h(\gamma)\varphi_0(y/\varepsilon)$, $\gamma \in \Gamma'$, and functions from $L_1^2(C_\varepsilon')$ on each cross-section of the channels are orthogonal to $\varphi_0(y/\varepsilon)$. Here and below the point $\gamma = \gamma(x) \in \Gamma$ is

defined by the cross-section of the channel through x. We put $\gamma_0 = \gamma(x_0)$, i.e. γ_0 is the point on the graph defined by the cross-section of the channel through x_0.

Consider the operator
$$K_{\lambda,\varepsilon} : L^2(C'_\varepsilon) \to L^2(\Omega_\varepsilon)$$

with kernel $K_{\lambda,\varepsilon}(x, x_0)$ defined as follows:

$$K_{\lambda,\varepsilon}(x, x_0) = \sum_{v \in V} \frac{1}{h(\varepsilon, \mu)} \sum_{1 \leq p \leq d(v)} a_{p,v} \widehat{\Psi}^{(\varepsilon)}_{p,v}(x, x_0), \qquad x_0 \in C'_\varepsilon, \ x \in \Omega_\varepsilon.$$

Here $h(\mu, \varepsilon)$ and $a_{p,v} = a_{p,v}(\gamma_0, \mu, \varepsilon)$ are functions defined in (28), and $\widehat{\Psi}^{(\varepsilon)}_{p,v}$ are defined by the scattering solutions $\Psi^{(\varepsilon)}_{p,v}$ of the problem in the spider domain $\Omega_{v,\varepsilon}$ in the following way. Let $\Omega^0_{v,\varepsilon}$ be the part of the spider domain $\Omega_{v,\varepsilon}$ which consists of the junction and parts of the adjacent channels up to the cylinders $C'_{j,\varepsilon}$. Let $\Omega'_{v,\varepsilon}$ ($\Omega^1_{v,\varepsilon}$) be a bigger domain which contains additionally the parts of the cylinders $C'_{j,\varepsilon}$ up to the cross-sections which correspond to points γ_j (the whole cylinders $C'_{j,\varepsilon}$, respectively). We put $\widehat{\Psi}^{(\varepsilon)}_{p,v} = \Psi^{(\varepsilon)}_{p,v}$ in $\Omega^0_{v,\varepsilon}$. We split the scattering solutions $\Psi^{(\varepsilon)}_{p,v}$ in the cylinders $C'_{j,\varepsilon}$ into the sum of two terms. The first term contains the main modes $\varphi_0(y/\varepsilon)e^{\pm i\sqrt{\mu}t}$, and the second one is orthogonal to $\varphi_0(y/\varepsilon)$ in each cross-section. We multiply the first term by the function $\theta_v(x, x_0)$ equal to one on $\Omega'_{v,\varepsilon}$ and equal to zero everywhere else on Ω_ε. We multiply the second term by an infinitely smooth cut-off function $\eta_v(x)$ equal to one on $\Omega^0_{v,\varepsilon}$ and equal to zero on Ω_ε outside $\Omega^1_{v,\varepsilon}$. In other terms,

$$\widehat{\Psi}^{(\varepsilon)}_{p,v}(x, x_0) = \theta_v(x, x_0)\Psi^{(\varepsilon)}_{p,v} + (\eta_v(x) - \theta_v(x, x_0))r^\varepsilon_{p,j}, \tag{29}$$

where $r^\varepsilon_{p,j}$ is defined in (27).

Recall that the representation $\Gamma = \cup\Gamma^{cut}_v$ depends on the choice of points $\gamma_s \in \Gamma'_s \subset \Gamma_s$. All these points are fixed arbitrarily except one: if $\gamma_0 \in \Gamma_j$ then γ_j is chosen to be equal to γ_0. This is the reason why θ_v depends on x_0 and η_v is x_0-independent.

We look for the parametrix (almost resolvent) of the operator H_ε, when (22) holds and $f \in L^2(C'_\varepsilon)$, in the form
$$F_{\lambda,\varepsilon} = K_{\lambda,\varepsilon}P_0 + R_{\lambda,\varepsilon}P_1,$$

where P_0, P_1 are projections on $L^2_0(C'_\varepsilon)$ and $L^2_1(C'_\varepsilon)$, respectively. It is not difficult to show that $\|R_{\lambda,\varepsilon}P_1\| = O(\varepsilon^2)$ and $H_\varepsilon F_{\lambda,\varepsilon} = I + F_{\lambda,\varepsilon}$, where $\|F_{\lambda,\varepsilon}\| = O(\varepsilon)$. This implies that $R_{\lambda,\varepsilon} = K_{\lambda,\varepsilon}P_0 + O(\varepsilon)$. The latter, together with Lemma 7, justifies Theorem 11.

6   The GC at λ near the threshold λ_0.

Theorem 11 and the remarks following the theorem indicate that the GC at each vertex when $\lambda - \lambda_0 = O(\varepsilon^2)$ is the Dirichlet/Neumann condition, i.e. the junctions of Ω_ε can be replaced by $k(v)$ Dirichlet and $d(v) - k(v)$ Neumann conditions at the edges of the channels adjacent to the junctions (after an appropriate orthogonal transformation). We are going to specify the choice between the Dirichlet and Neumann conditions. First, we would like to make four important

Remarks. 1) Classical Kirchhoff's GC corresponds to $k = d - 1$.

2) For any domain Ω_ε under consideration, if $\lambda - \lambda_0 = O(\varepsilon^2)$ and the Neumann boundary condition is imposed on $\partial\Omega_\varepsilon$ ($\lambda_0 = 0$ in this case) then the GC on the limiting graph is Kirchhoff's condition (see section 2).

3) It was proven in [21] that if $\lambda - \lambda_0 = O(\varepsilon^2)$ and the boundary condition on $\partial\Omega_\varepsilon$ is different from the Neumann condition, then the GC on the limiting graph is the Dirichlet condition ($k = d$) for generic domains Ω_ε. An example at the end of the next section illustrates this fact.

4) The theorem below states that Kirchhoff's GC condition on the limiting graph appears in the case of arbitrary boundary conditions on $\partial\Omega_\varepsilon$, if the operator H_ε has a ground state at $\lambda = \lambda_0$. The ground state at $\lambda = \lambda_0 = 0$ exists for an arbitrary domain Ω_ε, if the Neumann boundary condition is imposed on $\partial\Omega_\varepsilon$. The ground state at $\lambda = \lambda_0$ does not exist for generic domains Ω_ε in the case of other boundary conditions (see [21]).

Note that the GC is determined by the scattering matrix in the spider domain $\Omega_{v,\varepsilon}$, and this matrix does not depend on ε. Thus, when the GC is studied, it is enough to consider a spider ε-independent domain $\Omega = \Omega_{v,1}$. We shall omit the indices v and ε in H_ε, $C_{j,\varepsilon}$ when $\varepsilon = 1$.

Definition 12 *A ground state of the operator H in a spider domain Ω at $\lambda = \lambda_0$ is the function $\psi_0 = \psi_0(x)$, which is bounded, strictly positive inside Ω, satisfies the equation $(-\Delta - \lambda_0)\psi_0 = 0$ in Ω, and the boundary condition on $\partial\Omega$, and has the following asymptotic behavior at infinity*

$$\psi_0(x) = \varphi_0(y)\left[\rho_j + o(1)\right], \quad x \in C_j, \quad |x| \to +\infty, \tag{30}$$

where $\rho_j > 0$ and φ_0 is the ground state of the operator in the cross-sections of the channels.

Let us stress that we assume the strict positivity of ρ_j.

Let's consider the parabolic problem in a spider domain Ω_ε,

$$\frac{\partial u_\varepsilon}{\partial\tau} = \Delta u_\varepsilon, \quad u_\varepsilon(0,x) = \varphi(\gamma)\varphi_0(\frac{y}{\varepsilon}), \quad u_\varepsilon(\tau,x)|_{\partial\Omega_\varepsilon} = 0, \tag{31}$$

where $\gamma = \gamma(x) \in \Gamma$ is defined by the cross-section of the channel through the point x, function φ is continuous, compactly supported with a support outside of the junctions, and depends only on the longitudinal ("slow") variable t on each edge $\Gamma_j \subset \Gamma$. We shall denote the coordinate t on C_j and Γ_j by t_j. Let ω' be a compact in the cross-section ω of the channels C_j.

Theorem 13 . *Let Ω be a spider domain, the Dirichlet or Robin boundary condition be imposed at $\partial\Omega$, and let the operator H have a ground state at $\lambda = \lambda_0$. Then asymptotically, as $\varepsilon \to 0$, the solution of the parabolic problem (31) in Ω_ε has the following form*

$$u_\varepsilon(\tau,x) = e^{-\frac{\lambda_0\tau}{\varepsilon^2}}w_\varepsilon(\tau,x)\varphi_0(\frac{y}{\varepsilon}),$$

where the function w_ε converges uniformly in any region of the form $0 < c^{-1} < \tau < c$, $t_j(x) > \delta > 0$, $y \in \omega'$ to a function $w(\tau,\gamma)$ on the limiting graph Γ which satisfies the relations

$$\frac{\partial w}{\partial\tau} = \frac{\partial^2 w}{\partial t^2} \quad \text{on } \Gamma; \quad w \text{ is continuous at the vertex, } \sum_{j=1}^{d}\rho_j\frac{\partial w}{\partial t_j}(0) = 0. \tag{32}$$

Remarks. 1) Let's note that under the ground state condition, operator H_ε has no eigenvalues below λ_0. Otherwise, the eigenfunction with the eigenvalue $\lambda_{\min} < \lambda_0$ must be orthogonal to the ground state $\psi_0(x)$, and this contradicts the positivity of both functions.

2) The eigenvalues below λ_0 can exist if H does not have the ground state at λ_0. For instance, they definitely exist if one of the junctions is "wide enough" (in contrast to the O. Post condition [24]). The solution $u_\varepsilon(\tau,x)$ in this case has asymptotics different from the one stated in Theorem 13. In particular, if the function φ (see (31)) is positive, then

$$\varepsilon^2\ln u_\varepsilon(\tau,x) \to \lambda_{\min}, \quad \tau \to \infty.$$

What is more important, the total mass of the heat energy in this case is concentrated in an arbitrarily small, as $\varepsilon \to 0$, neighborhood of the junctions. The limiting diffusion process on Γ degenerates.

Proof. For simplicity, we shall assume that the Dirichlet boundary condition is imposed on $\partial\Omega_\varepsilon$. Obviously, the function $\psi_0\left(\frac{x}{\varepsilon}\right)$ is the ground state in the spider domain Ω_ε. In particular,

$$\varepsilon^2 \Delta\psi_0 + \lambda_0\psi_0 = 0; \quad \psi_0\left(\frac{x}{\varepsilon}\right) = \varphi_0(\frac{y}{\varepsilon})[\rho_j + o(1)], \quad x \in C_{j,\varepsilon}, \quad |x| \to +\infty; \quad \psi_0|_{\partial\Omega_\varepsilon} = 0 .$$

Put $u_\varepsilon(\tau,x) = \psi_0\left(\frac{x}{\varepsilon}\right) e^{-\frac{\lambda_0\tau}{\varepsilon^2}} w_\varepsilon(\tau,x)$. Then

$$\frac{\partial w_\varepsilon}{\partial\tau} = \Delta w_\varepsilon + \frac{2}{\varepsilon}\nabla\left(\ln\psi_0\left(\frac{x}{\varepsilon}\right)\right)\cdot\nabla w_\varepsilon,$$

$$w_\varepsilon(0,x) = \varphi(\gamma)\theta\left(\frac{x}{\varepsilon}\right), \quad \theta\left(\frac{x}{\varepsilon}\right) = \frac{1}{\rho_j} + o(1), \quad x \in C_j, \quad |x| \to +\infty. \tag{33}$$

We look for bounded solutions u_ε, w_ε of the parabolic problems. We do not need to impose boundary conditions on $\partial\Omega_\varepsilon$ on the function w_ε since the boundedness of w_ε implies that $u_\varepsilon = 0$ on $\partial\Omega_\varepsilon$. The parabolic problem (33) has a unique bounded solution (without boundary conditions on $\partial\Omega_\varepsilon$) since $\nabla_z \ln\psi_0(\cdot)$ is growing near $\partial\Omega_\varepsilon$. This growth of the coefficient in (33) does not allow the heat energy (or diffusion) to reach $\partial\Omega_\varepsilon$. The fundamental solution $q_\varepsilon = q_\varepsilon(\tau,x_0,x)$ of the problem (33) exists, is unique, and $\int_{\Omega_\varepsilon} q_\varepsilon(\tau,x_0,x)\,dx = 1$. This fundamental solution is the transition density of the Markov diffusion process $X_\tau^{(\varepsilon)} = (T_\tau^{(\varepsilon)}, Y_\tau^{(\varepsilon)})$ in Ω_ε with the generator $\tilde{\mathcal{H}}_\varepsilon = \Delta + \frac{2}{\varepsilon}\left(\nabla\ln\psi_0\left(\frac{x}{\varepsilon}\right),\nabla\right)$.

Let $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_\varepsilon$, and $q = q_\varepsilon$ when $\varepsilon = 1$. The coefficients of the operator $\tilde{\mathcal{H}}$ are singular at the boundary of the domain. However, the transition density $q(\tau,x_0,x)$ is not vanishing inside Ω. To be more exact, the Döblin condition holds, i.e. for any compact $\omega' \subset \omega$, there exist $\delta > 0$ such that for any channel C_j the following estimate holds

$$q(\tau,x_0,x) > \delta \quad \text{when} \quad T \geq \tau \geq 1, \quad x = (t,y), \quad x_0 = (t,y_0), \quad y,\ y_0 \in \omega'.$$

The operator $\tilde{\mathcal{H}} = \Delta + 2(\nabla\ln\psi_0(x),\nabla)$ has a unique (up to normalization) invariant measure. This measure has the density $\pi(x) = \psi_0^2(x)$. In fact, if $\tilde{\mathcal{H}} = \Delta + (\nabla A,\nabla)$ then $\tilde{\mathcal{H}}^* = \nabla - (\nabla A,\nabla) - (\Delta A)$, and one can easily check that $\tilde{\mathcal{H}}^* e^{A(x)} = 0$. If we put now $A(x) = 2\ln\psi_0(x)$, we get $\tilde{\mathcal{H}}^*\pi(x) = 0$.

When $t > \delta_0 > 0$ and $\varepsilon \to 0$ the transversal component $Y_\tau^{(\varepsilon)}$ and the longitudinal component $T_\tau^{(\varepsilon)}$ of the diffusion process in Ω_ε are asymptotically independent. The transversal component $Y_\tau^{(\varepsilon)}$ oscillates very fast and has asymptotically ($\varepsilon \to 0$) invariant measure $\frac{1}{\varepsilon}\varphi_0^2\left(\frac{x}{\varepsilon}\right)$. The latter follows from the Döblin condition. The longitudinal component has a constant diffusion with the drift which is exponentially small (of order $O(e^{-\frac{\gamma}{\varepsilon^2}})$, $\gamma > 0$) outside any neighborhood of the junction.

Under conditions above, one can apply (with minimal modifications) the fundamental averaging procedure by Freidlin-Wentzel (see [12]) which leads to the convergence (in law on each compact interval in τ) of the distribution of the process $X_\tau^{(\varepsilon)}$ to the distribution of the process on Γ with the generator $\frac{d^2}{dt^2}$ on the space of functions on Γ smooth outside of the vertex and satisfying the appropriate GC. The GC are defined by the limiting invariant measure. This limiting measure on Γ is equal (up to a normalization) to ρ_j on edges Γ_j. This leads to the GC (32) of the generalized Kirchhoff form. The proof is complete.

Theorem 14 . *Let operator H in a spider domain Ω with the Dirichlet or Robin condition at $\partial\Omega$ have a ground state at $\lambda = \lambda_0$, and let $\lambda = \lambda_0 + O(\varepsilon^2)$. Then the GC (17) has the generalized Kirchhoff form: ζ is continuous at the vertex v and $\sum_{j=1}^d \rho_j \frac{d\zeta}{dt_j}(0) = 0$.*

This statement follows immediately from Theorem 13 since it is already established that the GC has the Dirichlet/Neumann form.

7 Effective potential.

As it was already mentioned earlier, the GC (17) is λ-dependent. The following result allows one to reduce the original problem in Ω_ε to a Schrödinger equation on the limiting graph with arbitrary λ-independent GC and a λ-independent matrix potential. The potential depends on the choice of the GC. Only the lower part of the a.c. spectrum $\lambda_0 \leq \lambda \leq \lambda_1$ will be considered. It is assumed below that $\varepsilon = 1$, and the index ε is omitted everywhere.

Let $T(\lambda), \lambda \in [\lambda_0, \lambda_1]$, be the scattering matrix for a spider domain Ω, let $\lambda_{-N} \leq \lambda_{-N+1} \leq \cdots \leq \lambda_{-1} < \lambda_0$ be the eigenvalues of the discrete spectrum of H below the threshold λ_0. The function $T(\lambda)$ has analytic extension into the complex plane with the cut along $[0, \infty)$. It has poles at $\lambda = \lambda_{-N}, ..., \lambda_{-1}$. Let $m_{-N}, \cdots, m_{-1}$ be the corresponding residues (Hermitian $d \times d$ matrices). These residues contain complete information on the multiplicity of λ_j, $j = -N, \cdots - 1$ and on the exponential asymptotics of the eigenfunctions $\psi_j(x), |x| \to +\infty$.

Theorem 6 allows one to reduce the problem in Ω_ε to an equation for a function ζ on the limiting graph Γ with appropriate GC at the vertex. Consider the vector $\psi := \zeta^{(v)} = \zeta^{(v)}(t)$, $t \geq 0$, whose components are the restrictions of ζ to the edges of Γ. Note that the GC were formulated through the vector $\zeta^{(v)}$. Now we would also like to treat the equation for the function ζ on Γ as a linear system for the vector $\zeta^{(v)}$ on the half axis $t \geq 0$.

Theorem 15 . *There exists an effective fast decreasing matrix $d \times d$ potential $V(t)$ such that $V(t) = V^*(t)$, and the problem*

$$- \psi'' + [V(t) - \lambda_0 I]\psi = \lambda\psi, \quad t \geq 0, \quad \psi(0) = 0 \tag{34}$$

has the same spectral data on the interval $(-\infty, \lambda_1)$ as the original problem in Ω. The latter means that the scattering matrix $S(\lambda)$ of the problem (34) coincides with $T(\lambda)$ on the interval $[\lambda_0, \lambda_1]$, and the poles and residues of $S(\lambda)$ and $T(\lambda)$ are equal.

Remarks. 1) The potential is defined not uniquely.

2) The Dirichlet condition $\psi(0) = 0$ can be replaced by any fixed GC, say the Kirchhoff one (of course, with the different effective potential).

3) Different effective potentials appeared when explicitly solvable models were studied in our paper [20].

Proof. This statement is a simple corollary of the inverse spectral theory by Agranovich and Marchenko for 1-D matrix Schrödinger operators [1]. One needs only to show that $T(\lambda)$ can be extended to the semiaxis $\lambda > \lambda_1$ in such a way that the extension will satisfy all the conditions required by the Agranovich-Marchenko theory.

Example to the statements of Lemma 8 and Theorem 14. Consider the Schrödinger operator $H = -\frac{d^2}{dt^2} + v(t)$ on the whole axis with a potential $v(t)$ compactly supported on $[-1, 1]$. This operator may serve as a simplified version of the operator (34). The simplest explicitly solvable model from [20] also leads to the operator H. The GC at $t = 0$ for this explicitly solvable model are determined by the limit, as $\varepsilon \to 0$, of the solution of the equation $H_\varepsilon \psi_\varepsilon = f$, where $H_\varepsilon = -\frac{d^2}{dt^2} + \varepsilon^{-2} v\left(\frac{t}{\varepsilon}\right)$ and f is compactly supported and vanishing in a neighbourhood of $t = 0$. The solution ψ_ε is understood as L^2_{loc} limit of $(H_\varepsilon + i\mu)^{-1} f \in L^2$ as $\mu \to +0$.

Of course, $\lambda = 0$ is the bottom of the a.c. spectrum for H. If operator H does not have negative eigenvalues, then the equation $H\psi = 0$ has a unique (up to a constant factor) positive solution $\psi_0(t)$, which is not necessarily bounded. If this solution is linear outside $[-1, 1]$, then the limiting GC are the Dirichlet ones. This case is generic. If this solution is constant on one of the semiaxis, then the GC are the Dirichlet/Neumann conditions. Finally, if $\psi_0(t) = \rho_\pm$ for $\pm t \geq 1$, then we have the situation of Theorem 14: the ground state and the generalized Kirchhoff's GC.

One can get a nontrivial Kirchhoff's condition even in the case when H has a negative spectrum. It is sufficient to assume that $\lambda = 0$ is the eigenvalue (but not the minimal one) of the Neumann spectral problem for H on $[-1, 1]$.

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